Quiz 2 – Solutions

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1. Consider a matrix A, which we transform into the matrix

$$B = \begin{bmatrix} 0 & a & 0 & 0 & b \\ c & 0 & d & 0 & e \\ 0 & 0 & 0 & 1 & f \end{bmatrix}$$

by a sequence of elementary row operations. Assume $B = \text{rref}(A)$.

(a) What can we say about the constants a through f? What is the first column of A?

Solution: Because B is in reduced row echelon form:

- The leading 1 in the first nonzero row must occur in column 2, so $a = 1$.
- Leading 1s must move strictly to the right as we go down the rows, and all entries to the left of a pivot in that row must be 0. Hence in row 2 we must have $c = 0$ and the pivot must be in column 3, so $d = 1$.
- The third row already has a pivot 1 in column 4 (and zeros in column 4 above), which is compatible with the rref rules.
- Entries in nonpivot (free) columns may be arbitrary, so $b, e, f \in \mathbb{R}$.

Thus: $a = 1$, $c = 0$, $d = 1$, and b, e, f are arbitrary.

Since elementary row operations correspond to left–multiplication by an invertible matrix E, we have $B = EA$. The first column of B is

$$\begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so $E \cdot (\text{first column of } A) = \vec{0}$. As E is invertible, this implies the first column of A is also the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) We (temporarily) define the *rank* of a matrix to be the number of leading 1s in its reduced row echelon form. What is the rank of the matrix A?

Solution: In $B = \text{rref}(A)$ there are leading 1s in columns $2, 3, 4$ (one in each of the three nonzero rows). Hence $\text{rank}(A) = 3$.

2. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE and provide a counterexample.

(a) The solution set in $\mathbb{R}^3$ to two nontrivial linear equations in three unknowns is always a line or a plane in $\mathbb{R}^3$.

Solution: FALSE. Two planes in $\mathbb{R}^3$ can be parallel and hence disjoint, yielding *no* solutions.

Counterexample: $y + z = 0$ and $y + z = -1$ are parallel planes with empty intersection.

(b) Suppose A is a 16×25 matrix. If A has rank 14 and $\left[A \mid \vec{0}\right]$ is the augmented matrix of a linear system, then the system is consistent and has infinitely many solutions.

Solution: TRUE. The system is homogeneous, so $\vec{x} = \vec{0}$ satisfies $A\vec{x} = \vec{0}$, hence the system is consistent. In rref(A) there are 14 pivot columns, leaving $25 - 14 = 11$ free variables. Assigning arbitrary values to these free variables produces distinct solutions, so there are infinitely many solutions.