Quiz 1 – Solutions

Dr. Samir Donmazov

1. Consider the *Axiom of Parental Support*: If you get a "B" or better in this course, your parents will buy you a new car. Take the following definitions as given:

   Definition: An *"A" student* never gets a grade lower than "A−" in a given semester.

   Definition: A *"B" student* gets at most one grade lower than a "B" in a given semester.

   Definition: A *"C" student* gets no grade higher than "C" in a given semester.

   Decide if the following statements are Theorems. Justify each of your claims with either an argument or a counterexample.

   (a) If I am a "B" student, I will get a new car from my parents at the end of the semester.

   Solution: Not a Theorem. A "B" student is allowed at most one grade below "B." It is possible that the single grade below "B" is in this course (for instance, a "C" here). In that case, the axiom's condition of getting a "B or better in this course" is not met, so the car is not guaranteed.

   Counterexample: Grades (A, B, C, B, B) qualify as a "B" student, but the course grade is C, so no car is guaranteed.

   (b) If I am a "C" student, I will not get a new car from my parents at the end of the semester.

   Solution: Not a Theorem. Let P be "I get a B or better in this course" and Q be "My parents buy me a new car." The axiom states $P \Rightarrow Q$. A "C" student gets no grade higher than C, so in this course P is false (i.e., $\neg P$). However, from $P \Rightarrow Q$ and $\neg P$ we cannot conclude $\neg Q$ (that would be denying the antecedent).

   Counterexample: Receive a C in this course (so $\neg P$) and suppose the parents buy a car anyway (so Q). This does not violate the axiom $P \Rightarrow Q$, yet it contradicts the claim that a "C" student will not get a car. Hence, the statement is not a theorem.

   Logical note: The valid contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$, not $\neg P \Rightarrow \neg Q$.

2. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE and provide a counterexample.

   (a) The solution set in $\mathbb{R}^3$ to two linear equations in three unknowns is always a line in $\mathbb{R}^3$.

   Solution: FALSE. - Two independent linear equations in $\mathbb{R}^3$, two planes, can either intersect in a line, or in a plane (when one equation is a multiple of the other one) and do not intersect (when two planes are parallel).

   Counterexample: $y + z = 00$ and $y + z = -1$ are two parallel planes with no intersection.

(b) A plane in $\mathbb{R}^3$ has exactly one normal vector.

Solution: FALSE. If $\vec{n}$ is a normal vector to the plane, then any nonzero scalar multiple $c\vec{n}$ is also a normal vector. Thus, a plane has infinitely many normal vectors, all scalar multiples of each other.

Counterexample: For the plane $x + y + z = 1$, both $(1, 1, 1)$ and $(2, 2, 2)$ are normal vectors.