Math 217 Fall 2025
Quiz 25 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose $n \in \mathbb{N}$. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if

   > **Solution:** it preserves the Euclidean inner product, i.e.
   >
   > $$\vec{x} \cdot \vec{y} \;=\; T(\vec{x}) \cdot T(\vec{y}) \quad \text{for all } \vec{x}, \vec{y} \in \mathbb{R}^n.$$
   >
   > Equivalently, its standard matrix Q satisfies $Q^\top Q = I_n$.

   (b) Suppose $m \in \mathbb{N}$. An $m \times m$ matrix A is *orthogonal* if ...

   > **Solution:** its transpose is its inverse:
   >
   > $$A^\top A = I_m \quad (\text{equivalently, } AA^\top = I_m).$$
   >
   > This is equivalent to saying that A represents an inner-product–preserving linear map on $\mathbb{R}^m$.

   (c) Let $\mathfrak{B} = (v_1, \ldots, v_d)$ be a basis for the vector space V and let $v \in V$. The $\mathfrak{B}$-*coordinates* of v are ...

   > **Solution:** the unique scalars $a_1, \ldots, a_d$ such that
   >
   > $$v \;=\; a_1 v_1 + \cdots + a_d v_d.$$
   >
   > We collect them into the column vector $[v]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix}$.

2. (a) Suppose V is a vector space and $\mathcal{A} = (v_1, \ldots, v_n)$ is a maximal linearly independent subset of V. Show that $\mathcal{A}$ is a basis for V.

   > **Solution:** By definition, maximal linear independence means that no vector $v \in V \setminus \operatorname{Span}(\mathcal{A})$ can be added to $\mathcal{A}$ while preserving linear independence. If $\operatorname{Span}(\mathcal{A}) \neq V$, pick $w \in V \setminus \operatorname{Span}(\mathcal{A})$. Then $\mathcal{A} \cup \{w\}$ is linearly independent (since w is not in the span of $\mathcal{A}$), contradicting maximality. Hence $\operatorname{Span}(\mathcal{A}) = V$. Together with linear independence, this shows $\mathcal{A}$ is a basis.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(b) Suppose $m \in \mathbb{N}$ and A is an $m \times m$ matrix with columns $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^m$. Show that A is orthogonal if and only if its columns are orthonormal.

> **Solution:** Compute
>
> $$A^\top A = \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_m^\top \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} = \left(\vec{v}_i \cdot \vec{v}_j\right)_{1 \le i,j \le m}.$$
>
> Thus $A^\top A = I_m$ if and only if $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$, which means $\vec{v}_i \cdot \vec{v}_j = 1$ when $i = j$ and $\vec{v}_i \cdot \vec{v}_j = 0$, when $i \neq j$. Thus, the columns have unit length and are mutually orthogonal. Therefore, A is orthogonal $\iff$ its columns are orthonormal.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Suppose $n \in \mathbb{N}$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation. The standard matrix of T has columns that are orthonormal.

> **Solution: TRUE.** Let Q be the standard matrix of T. Orthogonality of T gives $Q^\top Q = I_n$. As in Q2(b), $Q^\top Q$ has (i, j)-entry $\vec{q}_i \cdot \vec{q}_j$, where $\vec{q}_j$ is the j-th column of Q. Hence $\vec{q}_i \cdot \vec{q}_j = \delta_{ij}$, so the columns are orthonormal.

(b) If $m \in \mathbb{N}$ and $O : \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal transformation, then O is injective.

> **Solution: TRUE.** Let Q be the matrix of O. Since O is orthogonal, $Q^\top Q = I_m$, so Q is invertible (indeed $Q^{-1} = Q^\top$). An invertible linear map is injective. Alternatively, if $O(\vec{x}) = \vec{0}$, then
>
> $$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x} = (O\vec{x}) \cdot (O\vec{x}) = \|\vec{0}\|^2 = 0,$$
>
> so $\vec{x} = \vec{0}$.