Math 217 Fall 2025
Quiz 17 – Solutions

Dr. Samir Donmazov

1. Complete[*] the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function T satisfying

   $$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(\alpha v) = \alpha\, T(v)$$

   for all $u, v \in V$ and all scalars α (equivalently, $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$ for all scalars α, β and $u, v \in V$).

   (b) A *subspace* of a vector space V is ...

   Solution: A subset $U \subseteq V$ that is itself a vector space under the operations inherited from V; equivalently,

   $$0_V \in U \quad \text{and} \quad \alpha u + \beta v \in U \;\text{ for all } u, v \in U \text{ and scalars } \alpha, \beta.$$

   (c) Suppose X and Y are sets. A function $f : X \to Y$ is *surjective* provided that ...

   Solution: For every $y \in Y$ there exists $x \in X$ with $f(x) = y$; i.e. $\text{im}(f) = Y$.

2. Suppose V and W are vector spaces and $T : V \to W$ is linear. Let $\{v_1, \dots, v_m\} \subset V$ be such that $\{T(v_1), \dots, T(v_m)\}$ is a basis of $\text{im}T$, and let $\{u_1, \dots, u_n\}$ be a basis of $\ker T$. Prove that $\{v_1, \dots, v_m, u_1, \dots, u_n\}$ is a linearly independent set in V.

   Solution: Assume

   $$a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_n u_n = 0_V.$$

   Apply T:

   $$a_1 T(v_1) + \cdots + a_m T(v_m) + b_1 T(u_1) + \cdots + b_n T(u_n) = 0_W.$$

   Since $u_j \in \ker T$, $T(u_j) = 0$ for all j. Hence

   $$a_1 T(v_1) + \cdots + a_m T(v_m) = 0_W.$$

[*] For full credit, please write out fully what you mean instead of using shorthand phrases.

But $\{T(v_1), \ldots, T(v_m)\}$ is a basis of im$T$, in particular a linearly independent set, so $a_1 = \cdots = a_m = 0$. The original relation then reduces to

$$b_1u_1 + \cdots + b_nu_n = 0_V,$$

and since $\{u_1, \ldots, u_n\}$ is a basis of $\ker T$, it is linearly independent; thus $b_1 = \cdots = b_n = 0$. Therefore all coefficients are zero, proving linear independence.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) If A is the standard matrix of a surjective linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, then

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Solution: FALSE. Surjectivity implies $\text{rank}(A) = 3$, so $\text{rref}(A)$ has three pivot columns and one free column. The non-pivot (free) column need not be the zero column. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

has rank 3 (so the associated T is surjective), and A is already in RREF; its fourth column is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) There is a linear transformation $T : \mathbb{R}^7 \to \mathcal{P}_{71}$ such that

$$\dim(\text{im}T) - \dim(\ker T) = 6.$$

Solution: FALSE. By Rank–Nullity, for any linear map with domain $\mathbb{R}^7$,

$$\dim(\text{im}T) + \dim(\ker T) = 7.$$

Hence

$$\dim(\text{im}T) - \dim(\ker T) = \big(\dim(\text{im}T)\big) - \big(7 - \dim(\text{im}T)\big) = 2\dim(\text{im}T) - 7.$$

The right-hand side is an odd integer (since $2\dim(\text{im}T)$ is even, subtracting 7 yields an odd number). It cannot equal 6, which is even. Equivalently, solving $2r - 7 = 6$ gives $r = 6.5$, impossible for an integer rank.