Math 217 Fall 2025

Quiz 16 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose A is an $m \times n$ matrix, the *transpose* of A is ...

   > **Solution:** The $n \times m$ matrix obtained by interchanging rows and columns of A; that is, if $A = (a_{ij})$, then
   > $$A^\top = (a_{ji}) \quad \text{so that} \quad (A^\top)_{ij} = a_{ji}.$$

   (b) Suppose V and W are vector spaces and $T\colon V \to W$ is a linear transformation. The *image* of T is ...

   > **Solution:** The set of all outputs of T, i.e.
   > $$\operatorname{im}(T) = \{\, T(v) \in W : \ v \in V \,\}.$$
   > It is a subspace of W.

   (c) Suppose U is a vector space and $u_1, \dots, u_n \in U$. The *span* of $(u_1, \dots, u_n)$ is ...

   > **Solution:** The set of all finite linear combinations of the u_i:
   > $$\operatorname{span}(u_1, \dots, u_n) = \Big\{\, a_1u_1 + \cdots + a_nu_n \ : \ a_1, \dots, a_n \in \mathbb{F} \Big\}.$$
   > It is the smallest subspace of U containing $\{u_1, \dots, u_n\}$.

2. Fix any ordered basis $(v_1, \dots, v_n)$ for V, and consider the map

$$\phi : \mathbb{R}^n \to V, \qquad \phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1v_1 + \cdots + a_nv_n.$$

   (a) Show that ϕ is a linear transformation.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: Let $x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $y = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{F}$. Then

$$\phi(\alpha x + \beta y) = \phi\left(\begin{bmatrix} \alpha a_1 + \beta b_1 \\ \vdots \\ \alpha a_n + \beta b_n \end{bmatrix}\right) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) v_i$$
$$= \alpha \sum_{i=1}^{n} a_i v_i + \beta \sum_{i=1}^{n} b_i v_i = \alpha\phi(x) + \beta\phi(y).$$

Thus ϕ is linear.

(b) Show that ϕ is an *isomorphism*.

Solution: Define $\psi : V \to \mathbb{R}^n$ by sending each $v \in V$ to its coordinate column relative to the basis $(v_1, \dots, v_n)$, i.e., if $v = \sum_{i=1}^{n} c_i v_i$, set $\psi(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then ψ is linear and

$$(\psi \circ \phi)\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = \psi\left(\sum_{i=1}^{n} a_i v_i\right) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \text{i.e.,} \quad \psi \circ \phi = \mathrm{Id}_{\mathbb{R}^n},$$
$$(\phi \circ \psi)(v) = \phi\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = \sum_{i=1}^{n} c_i v_i = v \quad \text{i.e.,} \quad \phi \circ \psi = \mathrm{Id}_V .$$

Hence ϕ is bijective with inverse ψ, so ϕ is an isomorphism.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) If V is a vector space and $\mathcal{S}$ is a finite list of vectors in V such that $\vec{0}$ is on the list, then $\mathcal{S}$ is linearly dependent.

Solution: TRUE. If 0_V is in the list, then $1 \cdot 0_V + 0 \cdot (\text{others}) = 0_V$ is a nontrivial linear relation, so the list is linearly dependent.

(b) Any four vectors in $\mathbb{R}^3$ are linearly dependent.

Solution: TRUE. $\dim(\mathbb{R}^3) = 3$, so any list containing more than three vectors in $\mathbb{R}^3$ must be linearly dependent.

Equivalently, by the Rank–Nullity Theorem, for a linear map represented by a $3 \times n$ matrix, the rank cannot exceed 3. Hence, if $n > 3$, the nullity (dimension of the kernel) is positive, meaning the columns are linearly dependent. (For example, in a 3×3 matrix, the kernel is trivial precisely when the columns are independent.)